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The intersection of the shockwave of hydromechanical parameters (computation aspects)

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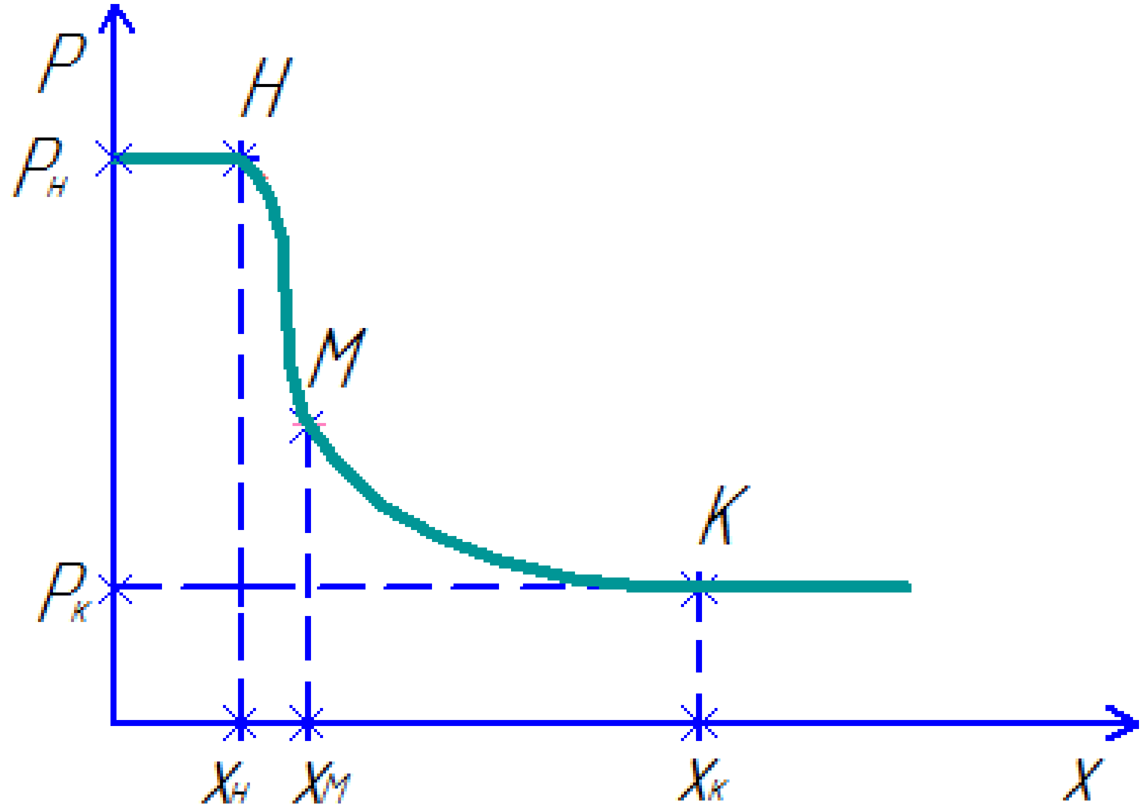

Fig.1.
Change in pressure when passing through the shockwave
(incident shockwave)
M - inflection point *P(x)*
H,K - start and end points of shockwave

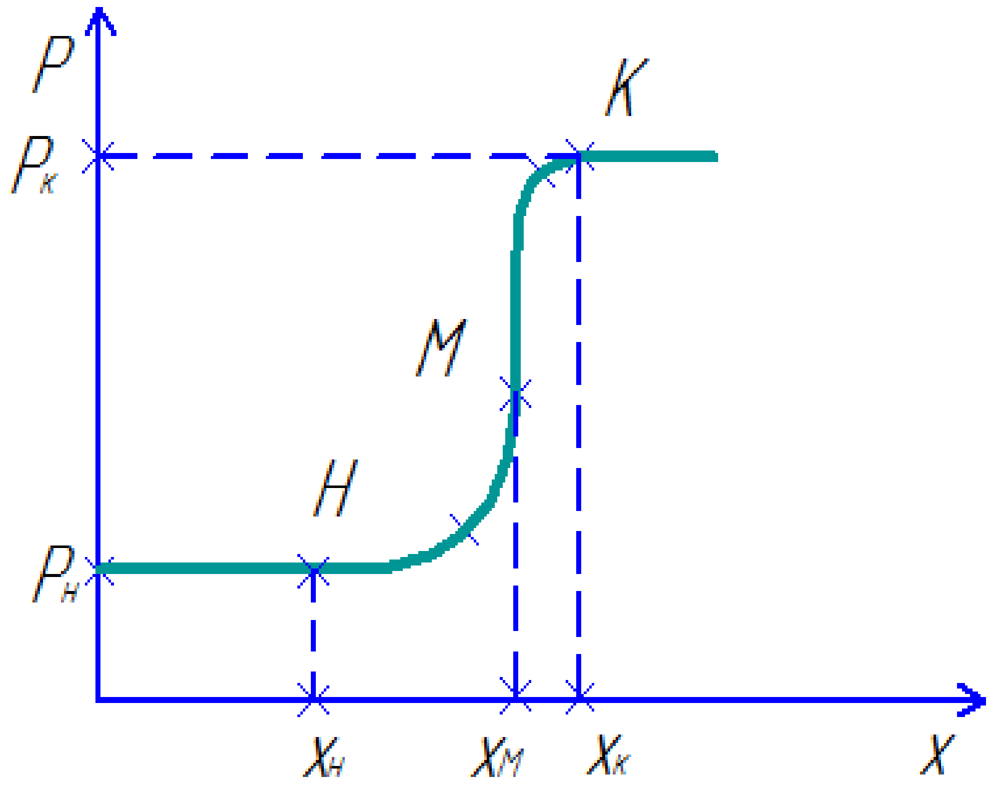

Fig.2
Change in pressure when passing through the ballistic wave
Pн, Pк - pressure before and after the shockwave
M - inflection point *P(x)*

You can see that when crossing the shock wave curves *Δx(x)* have the form :

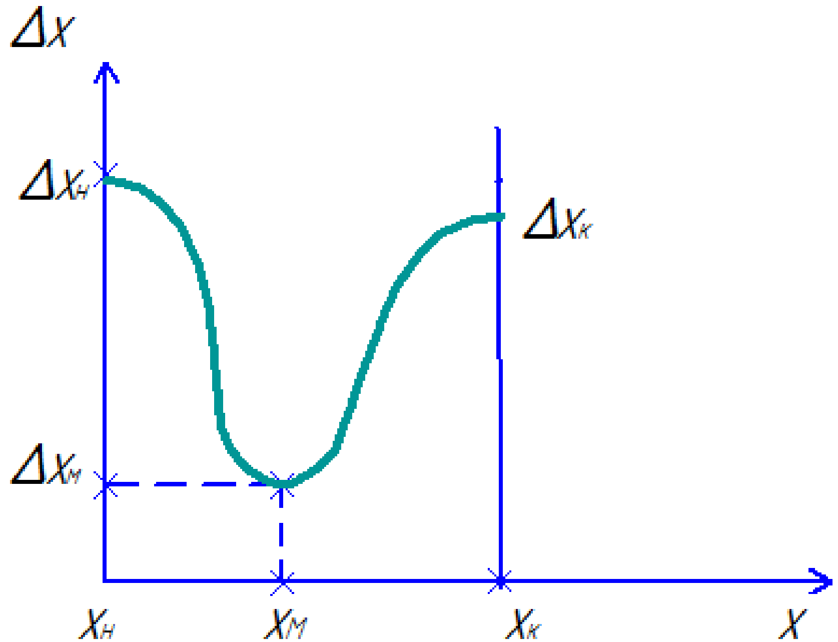

Fig.3

A qualitative picture of the integration steps changes along the *x* coordinate

$$\Delta x_H \neq \Delta x_K; \quad |x_M - x_H| \neq |x_K - x_M|$$

$$x_{i+1} = x_i + \Delta x_i; \quad \Delta x_i = f(x_i).$$

Transfer the beginning of the *x*-axis at the point M.
On the ascending branch of the curve *Δ x (x)* using equation

$$\frac{d\Delta x}{dx} = a(z^n - z)\cdot\varphi(x),$$ where

$$z = \frac{\Delta x - \Delta x_M}{\Delta x_K - \Delta x_M}$$

$$\varphi(x) = x^{-\alpha}(1+\beta x)$$

$$n > 0,\ \ \alpha > 0,\ \ \alpha < n,\ \ \beta > 0$$

From here

$$z = (1 - e^{-A\cdot F(x)})^{\frac{1}{1-n}}$$

$$\Delta x = \Delta x_M + (\Delta x_K - \Delta x_M)\cdot z$$

$$A = \frac{a(1-n)}{(1-\alpha)(\Delta x_K - \Delta x_M)}$$

$$F(x) = x^{1-\alpha}(1+\beta\frac{1-\alpha}{2-\alpha}x)$$

For the descending branch of *x (x)* we have the equation :

$$\frac{d\Delta x}{dx} = -a(\tilde{z}^n - \tilde{z}) \cdot \varphi(-x),$$

where $\tilde{z} = \dfrac{\Delta x - \Delta x_M}{\Delta x_H - \Delta x_M}$

We can write:

$$\frac{d\tilde{z}}{\tilde{z}^n - \tilde{z}} = \frac{a}{\Delta x_H - \Delta x_M} \cdot \varphi(-x) d(-x),$$

From here

$$\tilde{z} = (1 - e^{-A \cdot F(-x)})^{\frac{1}{1-n}}$$

$$\Delta x = \Delta x_M + (\Delta x_H - \Delta x_M) \cdot \tilde{z}$$

If $\Delta x_H = \Delta x_K = 0{,}06$

$\Delta x_M = 0{,}0005$

$n = 0{,}2; \quad \alpha = 0{,}1; \quad a = 1; \quad \beta = 100,$ then we have a curve *Δx(x)*

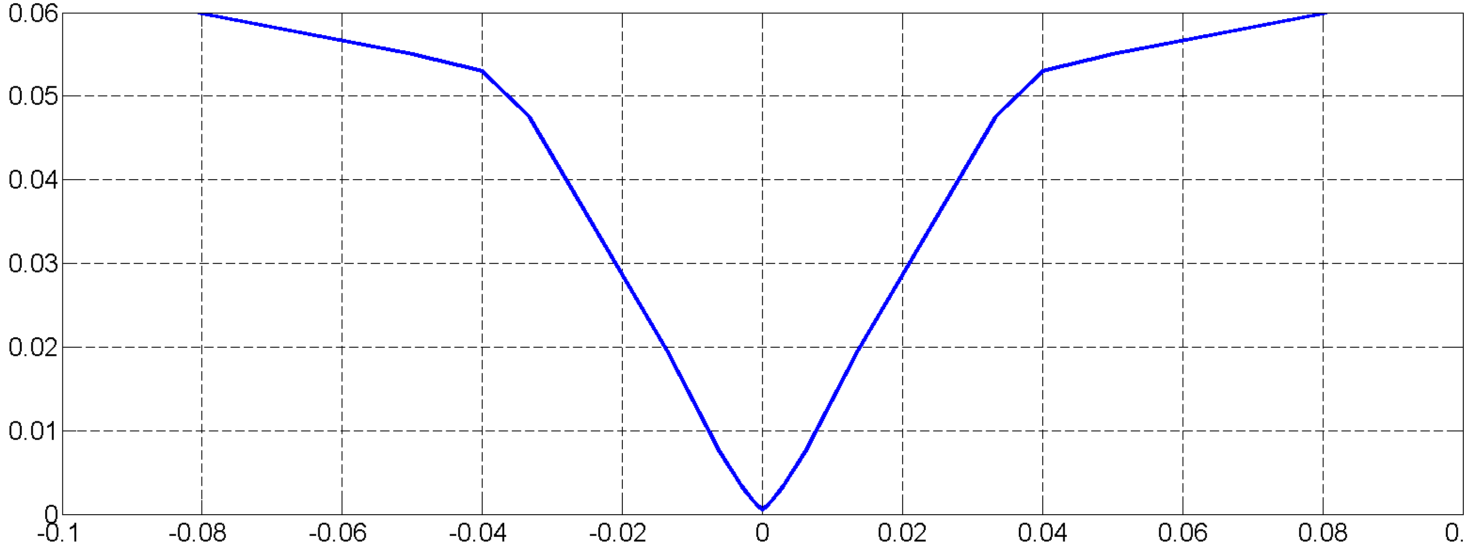

Fig.4
Step change *Δx* function coordinate *x*
(symmetric case)

Step size Δx_i depending on x_i

Table 1

i	0	1	2	3	4	5	6	7
x_i	0	0.0005	0.00134444	0.002926	0.006171	0.01364	0.03322	0.08065
Δx_i	0.005	0.0008444	0.001582	0.003245	0.007465	0.01958	0.04743	0.05995

Ratio of steps $\dfrac{\Delta x_{i+1}}{\Delta x_i}$

Table 2

i	0	1	2	3	4	5	6
$\Delta x_{i+1}/\Delta x_i$	1.6888	1.8739	2.0508	2.3006	2.6232	2.4218	1.2641

If $\Delta x_H = \Delta x_K = 0{,}06$

$\Delta x_M = 0{,}002$

$n = 0{,}2; \quad \alpha = 0{,}1; \quad a = 1; \quad \beta = 100,$ then we have a curve *Δx(x)*

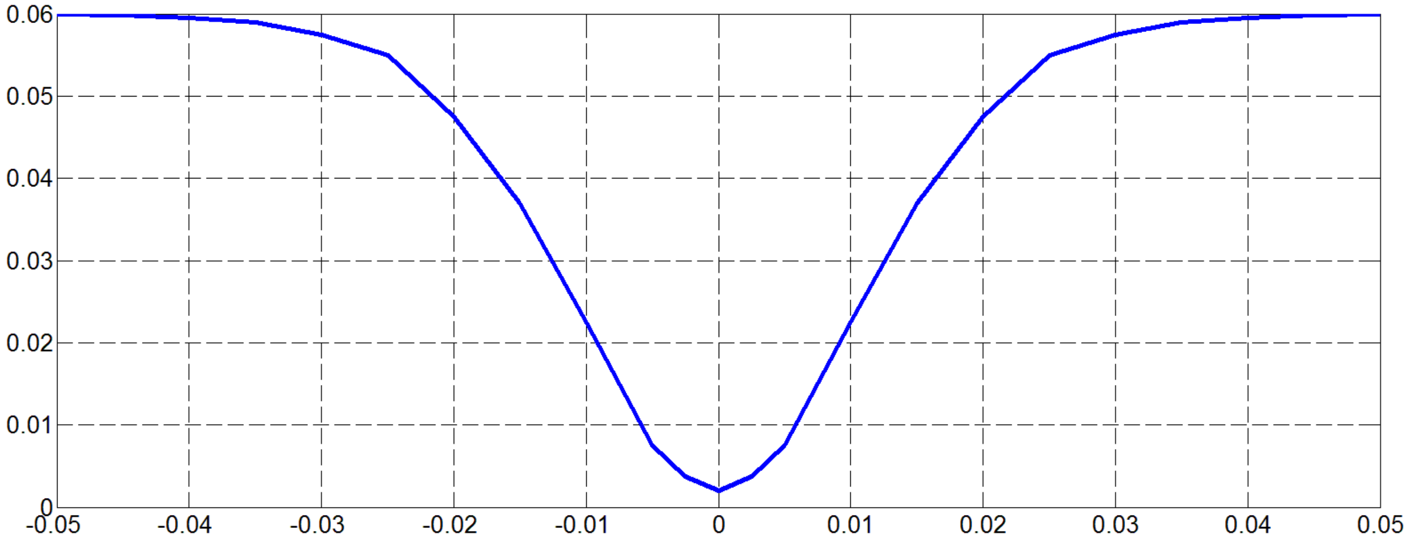

Fig.5

Step change ∆x function coordinate x

(symmetric case)

Finding the gradients algorithm

$$\frac{\partial \tilde{f}_i}{\partial x} \cdot \sum_{\theta=-(\frac{N}{2}-1)}^{\frac{N}{2}} \Delta x_{i+\theta} = \sum_{m=1}^{N/2} \left[a_{i+m}(f_{i+m} - f_i) - a_{i-m}(f_{i-m} - f_i) \right] \quad (14)$$

Here

$$f_{i+m} = f_i + \frac{\partial f_i}{\partial x} \sum_{\theta=1}^{m} \Delta x_{i+\theta} + \frac{1}{2} \frac{\partial^2 f_i}{\partial x^2} \left(\sum_{\theta=1}^{m} \Delta x_{i+\theta} \right)^2 + \frac{1}{6} \frac{\partial^3 f_i}{\partial x^3} \left(\sum_{\theta=1}^{m} \Delta x_{i+\theta} \right)^3 + \ldots \quad (15)$$

$$f_{i-m} = f_i - \frac{\partial f_i}{\partial x} \sum_{\theta=-(m-1)}^{0} \Delta x_{i+\theta} + \frac{1}{2} \frac{\partial^2 f_i}{\partial x^2} \left(\sum_{\theta=-(m-1)}^{0} \Delta x_{i+\theta} \right)^2 - \frac{1}{6} \frac{\partial^3 f_i}{\partial x^3} \left(\sum_{\theta=-(m-1)}^{0} \Delta x_{i+\theta} \right)^3 + \ldots \quad (16)$$

$\frac{\partial \tilde{f}_i}{\partial x}$ - approximate value $\frac{\partial f_i}{\partial x}$ in *i-th* point.

N- the total number of points on the left and right of the *i*-th. The value of the parameter N is determined based on the convergence of computing process when solving the equations of hydromechanics.

Let N=2. Then

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[a_{i+1}(f_{i+1} - f_i) - a_{i-1}(f_{i-1} - f_i) \right] \quad (17)$$

Expanding variables $f_{i\pm1}$ in the Taylor series, we obtain:

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{a_{i+1}\Delta x_{i+1} + a_{i-1}\Delta x_i}{\Delta x_i + \Delta x_{i+1}} \frac{\partial f_i}{\partial x} + \frac{a_{i+1}\Delta x^2{}_{i+1} - a_{i-1}\Delta x^2{}_i}{\Delta x_i + \Delta x_{i+1}} \frac{1}{2} \frac{\partial^2 f_i}{\partial x^2} + \frac{a_{i+1}\Delta x^3{}_{i+1} - a_{i-1}\Delta x^3{}_i}{\Delta x_i + \Delta x_{i+1}} \frac{1}{6} \frac{\partial^3 f_i}{\partial x^3} + \dots \quad (18)$$

Hence, we find:

$$a_{i+1}\Delta x_{i+1} + a_{i-1}\Delta x_i = \Delta x_i + \Delta x_{i+1}$$

$$a_{i+1}\Delta x^2{}_{i+1} - a_{i-1}\Delta x^2{}_i = 0$$

So that

$$a_{i+1} = \frac{\Delta x_i}{\Delta x_{i+1}}; \quad a_{i-1} = \frac{\Delta x_{i+1}}{\Delta x_i} \qquad \frac{\partial f_i}{\partial x} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[\frac{\Delta x_i}{\Delta x_{i+1}}(f_{i+1} - f_i) - \frac{\Delta x_{i+1}}{\Delta x_i}(f_{i-1} - f_i) \right]$$

Assessment of accuracy result from expansion (18):

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{\partial f_i}{\partial x} + \frac{\Delta x_i \cdot \Delta x_{i+1}}{6} \frac{\partial^3 f_i}{\partial x^3}$$

Let $N=4$. Then from (14) we find :

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{1}{\sum_{\theta=-1}^{2} \Delta x_{i+\theta}} \left[a_{i+2}(f_{i+2} - f_i) + a_{i+1}(f_{i+1} - f_i) - a_{i-2}(f_{i-2} - f_i) - a_{i-1}(f_{i-1} - f_i) \right]$$

Expanding variables $f_{i\pm 2}$ and $f_{i\pm 1}$ in the Taylor series, we obtain:

$$\begin{aligned}
\frac{\partial \tilde{f}_i}{\partial x} = {} & \frac{a_{i+2}(\Delta x_{i+1} + \Delta x_{i+2}) + a_{i+1}\Delta x_{i+1} + a_{i-2}(\Delta x_{i-1} + \Delta x_i) + a_{i-1}\Delta x_i}{\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1} + \Delta x_{i+2}} \frac{\partial f_i}{\partial x} + \\
& + \frac{a_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})^2 + a_{i+1}\Delta x^2{}_{i+1} - a_{i-2}(\Delta x_{i-1} + \Delta x_i)^2 - a_{i-1}\Delta x^2{}_i}{\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1} + \Delta x_{i+2}} \frac{1}{2} \frac{\partial^2 f_i}{\partial x^2} + \\
& + \frac{a_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})^3 + a_{i+1}\Delta x^3{}_{i+1} + a_{i-2}(\Delta x_{i-1} + \Delta x_i)^3 + a_{i-1}\Delta x^3{}_i}{\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1} + \Delta x_{i+2}} \frac{1}{6} \frac{\partial^3 f_i}{\partial x^3} + \\
& + \frac{a_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})^4 + a_{i+1}\Delta x^4{}_{i+1} - a_{i-2}(\Delta x_{i-1} + \Delta x_i)^4 - a_{i-1}\Delta x^4{}_i}{\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1} + \Delta x_{i+2}} \frac{1}{24} \frac{\partial^4 f_i}{\partial x^4} + \\
& + \frac{a_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})^5 + a_{i+1}\Delta x^5{}_{i+1} + a_{i-2}(\Delta x_{i-1} + \Delta x_i)^5 + a_{i-1}\Delta x^5{}_i}{\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1} + \Delta x_{i+2}} \frac{1}{120} \frac{\partial^5 f_i}{\partial x^5} \ldots
\end{aligned} \quad (23)$$

Hence, we find:

$$a_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})+a_{i+1}\Delta x_{i+1}+a_{i-2}(\Delta x_{i-1}+\Delta x_i)+a_{i-1}\Delta x_i=\Delta x_{i-1}+\Delta x_i+\Delta x_{i+1}+\Delta x_{i+2}$$
$$a_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^2+a_{i+1}\Delta x^2{}_{i+1}-a_{i-2}(\Delta x_{i-1}+\Delta x_i)^2-a_{i-1}\Delta x^2{}_i=0$$
$$a_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^3+a_{i+1}\Delta x^3{}_{i+1}+a_{i-2}(\Delta x_{i-1}+\Delta x_i)^3+a_{i-1}\Delta x^3{}_i=0$$
$$a_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^4+a_{i+1}\Delta x^4{}_{i+1}-a_{i-2}(\Delta x_{i-1}+\Delta x_i)^4-a_{i-1}\Delta x^4{}_i=0$$

If the main determinant of the system Δ_{04} is nonzero, then for the coefficients $a_{i+\theta}(\theta=1,2,3,4)$ we obtain

$$a_{i+\theta}=\frac{\Delta_{\theta 4}}{\Delta_{04}}$$

Wherein $$\frac{\partial\tilde{f}_i}{\partial x}=\frac{\partial f_i}{\partial x}+\theta(\Delta x^4{}_{ýô})\frac{\partial^5 f_i}{\partial x^5},$$

where

$$\theta(\Delta x^4{}_{ýô})=\frac{\Delta_{14}(\Delta x_{i+1}+\Delta x_{i+2})^5+\Delta_{24}\Delta x^5{}_{i+1}+\Delta_{34}(\Delta x_{i-1}+\Delta x_i)^5+\Delta_{44}\Delta x^5{}_i}{120\cdot\Delta_{04}(\Delta x_{i-1}+\Delta x_i+\Delta x_{i+1}+\Delta x_{i+2})}$$

Similarly coefficients $a_{i+\theta}$ may be found in any other N.

The algorithm for calculating the values of second order partial

We use the expression

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{cp} = \sum_{m=1}^{N/2} b_{i+m}(f_{i+m} - f_i) + \sum_{m=1}^{N/2} b_{i-m}(f_{i-m} - f_i)$$

Here, at the same value N, as in the previous case, the number of coefficients $b_{i\pm m}$ must be one greater. Only in this case under dividing by $(\Delta x_i)^2_{\text{cp}}$ we can get the same order of accuracy, which when divided by

$$\sum_{\theta=-(\frac{N}{2}-1)}^{\frac{N}{2}} \Delta x_{i+\theta}$$

We use the minimally asymmetrical difference scheme, taking a direction to minimize the $\Delta x(x)$ $\frac{N}{2}$ points, and in the opposite direction $\frac{N}{2}+1$

So if N=2 we have:

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{cp} = b_{i+1}(f_{i+1} - f_i) + b_{i-1}(f_{i-1} - f_i) + b_{i-2}(f_{i-2} - f_i),$$

where $(\Delta x_i)_{cp} = \frac{\Delta x_i + \Delta x_{i+1}}{2}$

Expanding variables $f_{i\mp1}$ and f_{i-2} in the Taylor series, we obtain:

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{cp} = \left[b_{i+1}\Delta x_{i+1} - b_{i-1}\Delta x_i - b_{i-2}(\Delta x_{i-1} + \Delta x_i)\right]\cdot\frac{\partial f_i}{\partial x} +$$

$$+ \left[b_{i+1}\Delta x^2{}_{i+1} + b_{i-1}\Delta x^2{}_i + b_{i-2}(\Delta x_{i-1} + \Delta x_i)^2\right]\cdot\frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} +$$

$$+ \left[b_{i+1}\Delta x^3{}_{i+1} - b_{i-1}\Delta x^3{}_i - b_{i-2}(\Delta x_{i-1} + \Delta x_i)^3\right]\cdot\frac{1}{6}\frac{\partial^3 f_i}{\partial x^3} +$$

$$+ \left[b_{i+1}\Delta x^4{}_{i+1} + b_{i-1}\Delta x^4{}_i + b_{i-2}(\Delta x_{i-1} + \Delta x_i)^4\right]\cdot\frac{1}{24}\frac{\partial^4 f_i}{\partial x^4}$$

Hence, we find:

$$b_{i+1}\Delta x_{i+1} - b_{i-1}\Delta x_i - b_{i-2}(\Delta x_{i-1} + \Delta x_i) = 0$$

$$b_{i+1}\Delta x^2{}_{i+1} + b_{i-1}\Delta x^2{}_i + b_{i-2}(\Delta x_{i-1} + \Delta x_i)^2 = 2(\Delta x_i)^2_{cp}$$

$$b_{i+1}\Delta x^3{}_{i+1} - b_{i-1}\Delta x^3{}_i - b_{i-2}(\Delta x_{i-1} + \Delta x_i)^3 = 0$$

The main determinant of this system Δ_{03} is:

$$\Delta_{03} = -\Delta x_i \cdot \Delta x_{i+1} \cdot \Delta x_{i-1}(\Delta x_{i-1} + \Delta x_i)(\Delta x_i + \Delta x_{i+1})(\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1})$$

and nonzero.

To calculate the values of the coefficients $b_{i\mp1}$ and b_{i-2} we find formula :

$$b_{i+1} = \frac{(\Delta x_i + \Delta x_{i+1})(\Delta x_{i-1} + 2\Delta x_i)}{2 \cdot \Delta x_{i+1}(\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1})}$$

$$b_{i-1} = \frac{(\Delta x_i + \Delta x_{i+1})(\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1})}{2 \cdot \Delta x_i \cdot \Delta x_{i-1}}$$

$$b_{i-2} = -2\frac{(\Delta x_i)^2_{cp}(\Delta x_i - \Delta x_{i+1})}{\Delta x_{i-1}(\Delta x_{i-1} + \Delta x_i)(\Delta x_{i-1} + \Delta x_i + \Delta x_{i+1})}$$

For $\Delta x_i \to \Delta x = const$

coefficient $b_{i-2} \to 0$, and the minimum asymmetric difference scheme becomes symmetric.

To assess the accuracy of calculation $\frac{\partial^2 \tilde{f}_i}{\partial x^2}$ for $N=2$ we have the equality:

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2} = \frac{\partial^2 f_i}{\partial x^2} + \theta(\Delta x^2_{э\,cp}) \cdot \frac{\partial^4 f_i}{\partial x^4}$$

$$\theta(\Delta x_э^2)_{cp} = \frac{b_{i+1}\Delta x^4{}_{i+1} + b_{i-1}\Delta x^4{}_i + b_{i-2}(\Delta x_{i-1} + \Delta x_i)^4}{24 \cdot (\Delta x_i^2)_{cp}}$$

For $N=4$ for the descending branch of the curve $\Delta x(x)$ we have:

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{cp} = b_{i+2}(f_{i+2}-f_i)+b_{i+1}(f_{i+1}-f_i)+b_{i-1}(f_{i-1}-f_i)+b_{i-2}(f_{i-2}-f_i)+b_{i-3}(f_{i-3}-f_i)$$

Expanding variables $b_{i\mp2}, b_{i\mp1}, b_{i-3}$ in the Taylor series for $\frac{\partial^2 \tilde{f}_i}{\partial x^2}$ we obtain:

$$\begin{aligned}
\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{cp} = {} & \left[b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})+b_{i+1}\Delta x_{i+1}-b_{i-1}\Delta x_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)-b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)\right]\cdot\frac{\partial f_i}{\partial x}+ \\
& +\left[b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^2+b_{i+1}\Delta x^2{}_{i+1}+b_{i-1}\Delta x^2{}_i+b_{i-2}(\Delta x_{i-1}+\Delta x_i)^2+b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^2\right]\cdot\frac{1}{2}\frac{\partial^2 f_i}{\partial x^2}+ \\
& +\left[b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^3+b_{i+1}\Delta x^3{}_{i+1}-b_{i-1}\Delta x^3{}_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)^3-b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^3\right]\cdot\frac{1}{6}\frac{\partial^3 f_i}{\partial x^3}+ \\
& +\left[b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^4+b_{i+1}\Delta x^4{}_{i+1}+b_{i-1}\Delta x^4{}_i+b_{i-2}(\Delta x_{i-1}+\Delta x_i)^4+b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^4\right]\cdot\frac{1}{24}\frac{\partial^4 f_i}{\partial x^4}+ \\
& +\left[b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^5+b_{i+1}\Delta x^5{}_{i+1}-b_{i-1}\Delta x^5{}_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)^5-b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^5\right]\cdot\frac{1}{120}\frac{\partial^5 f_i}{\partial x^5}+ \\
& +\left[b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^6+b_{i+1}\Delta x^6{}_{i+1}+b_{i-1}\Delta x^6{}_i+b_{i-2}(\Delta x_{i-1}+\Delta x_i)^6+b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^6\right]\cdot\frac{1}{720}\frac{\partial^6 f_i}{\partial x^6}
\end{aligned}$$

Hence, we find:

$$b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})+b_{i+1}\Delta x_{i+1}-b_{i-1}\Delta x_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)-b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)=0$$

$$b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^2+b_{i+1}\Delta x^2{}_{i+1}+b_{i-1}\Delta x^2{}_i+b_{i-2}(\Delta x_{i-1}+\Delta x_i)^2+b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^2=2(\Delta x_i)^2_{ep}$$

$$b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^3+b_{i+1}\Delta x^3{}_{i+1}-b_{i-1}\Delta x^3{}_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)^3-b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^3=0$$

$$b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^4+b_{i+1}\Delta x^4{}_{i+1}+b_{i-1}\Delta x^4{}_i+b_{i-2}(\Delta x_{i-1}+\Delta x_i)^4+b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^4=0$$

$$b_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})^5+b_{i+1}\Delta x^5{}_{i+1}-b_{i-1}\Delta x^5{}_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)^5-b_{i-3}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^5=0$$

If the main determinant of this system of equations Δ_{05} is different from zero, then coefficient $b_{i\mp2}, b_{i\mp1}, b_{i-3}$ can be defined as the ratio of $b_{i+\theta}=\frac{\Delta_{\theta4}}{\Delta_0}$,
where $\Delta_{\theta4}$ obtained from Δ_{04} replacing θ-th column by right parts (θ=2, 1, -1, -2, -3)
Wherein

$$\frac{\partial^2\tilde{f}_i}{\partial x^2}=\frac{\partial^2 f_i}{\partial x^2}+\theta(\Delta x^4_{i\,ýô})\cdot\frac{\partial^6 f_i}{\partial x^6}$$

where

$$\theta(\Delta x^4_{i\,ýô})=\frac{\Delta_{24}(\Delta x_{i+1}+\Delta x_{i+2})^6+\Delta_{14}\Delta x^6{}_{i+1}+\Delta_{-14}\Delta x^6{}_i+\Delta_{-24}(\Delta x_{i-1}+\Delta x_i)^6+\Delta_{-34}(\Delta x_{i-2}+\Delta x_{i-1}+\Delta x_i)^6}{720\cdot\Delta_{04}\cdot(\Delta x_i^2)_{ñð}}$$

If it is the case of a settlement when $\Delta x_{н} > \Delta x_{к}$ or if $|x_M - x_H| < |x_K - x_M|$ then the point M refer to the descending branch of the curve $\Delta x(x)$.

If $x_H < x_K$ or if $|x_M - x_H| > |x_K - x_M|$

then the point M refer to the ascending branch of the curve $\Delta x(x)$.

In the symmetric case, both ascending and descending branches of the curve $\Delta x(x)$ we finish in closest to M points $i+1$ and $i-1$, respectively.

Calculation of partial quantities at M we provide under additional algorithm for symmetric case.

Let in i-th point (point M) following equalities $\Delta x_i = \Delta x_{i+1}; \quad \Delta x_{i-1} = \Delta x_{i+2};$

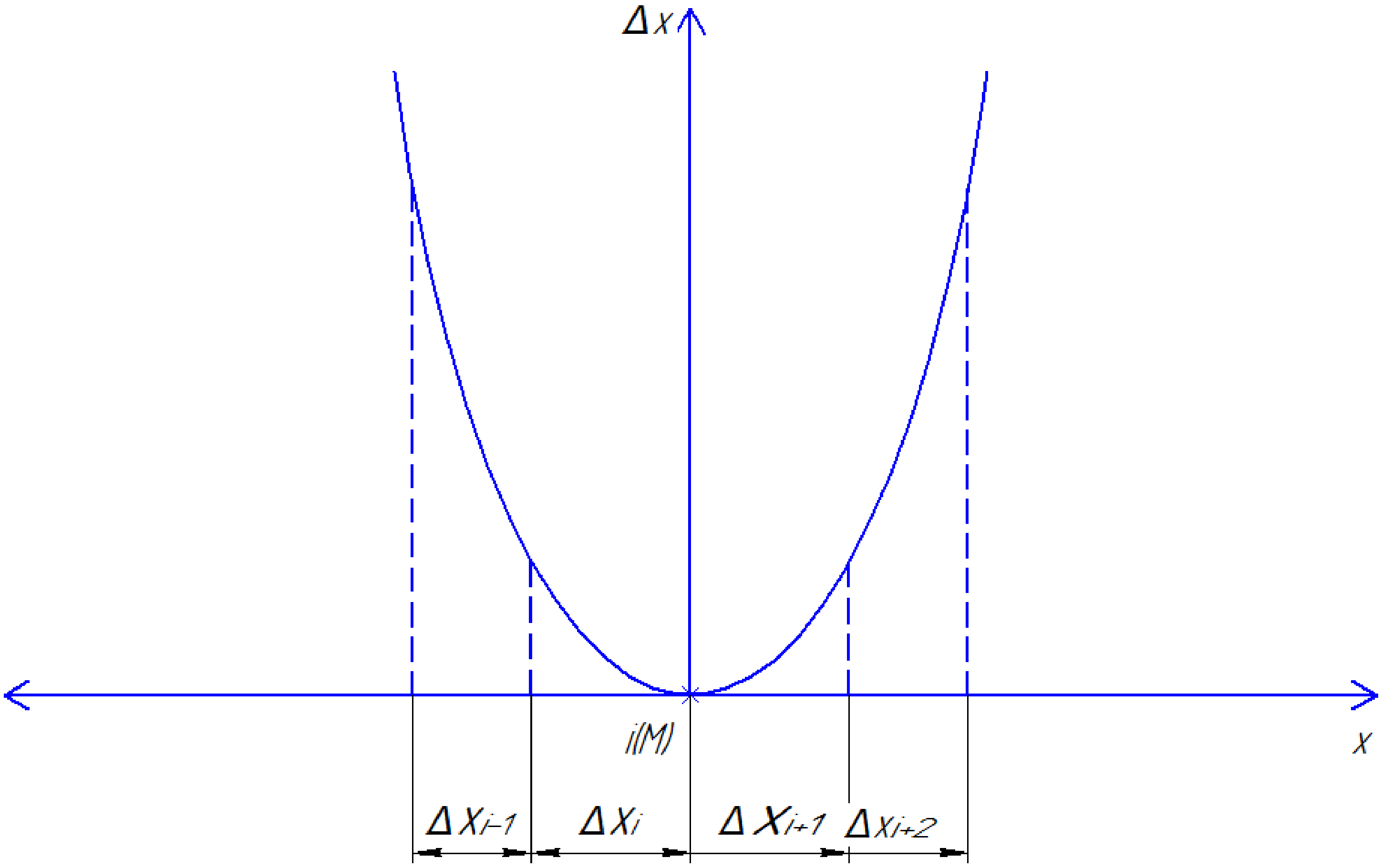

Fig. 6

Symmetric variables steps $\Delta x(x)$ change

(for the minimum point)

Using the symmetry ratio as in the calculation of the gradient and the second partial derivatives. To calculate the first derivative, we have:

$$\frac{\partial \tilde{f}_i}{\partial x} 2 \sum_{\theta=-(\frac{N}{2}-1)}^{0} \Delta x_{i+\theta} = \sum_{m=1}^{N/2} \left[a_{i+m}(f_{i+m} - f_i) - a_{i-m}(f_{i-m} - f_i) \right]$$

For $N=2$ we obtain:

$$\frac{\partial \tilde{f}_i}{\partial x} \cdot 2\Delta x_i = a_{i+1}(f_{i+1} - f_i) - a_{i-1}(f_{i-1} - f_i)$$

Expanding variables $f_{i\mp1}$ in the Taylor series, we obtain:

$$\frac{\partial \tilde{f}_i}{\partial x} \cdot 2\Delta x_i = (a_{i+1}\Delta x_{i+1} + a_{i-1}\Delta x_i)\frac{\partial f_i}{\partial x} + (a_{i+1}\Delta x^2{}_{i+1} - a_{i-1}\Delta x^2{}_i)\frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} + (a_{i+1}\Delta x^3{}_{i+1} + a_{i-1}\Delta x^3{}_i)\frac{1}{6}\frac{\partial^3 f_i}{\partial x^3}$$

Hence, we find:

$$a_{i+1}\Delta x_{i+1} + a_{i-1}\Delta x_i = 2\Delta x_i$$

$$a_{i+1}\Delta x^2{}_{i+1} - a_{i-1}\Delta x^2{}_i = 0$$

Considering that $\Delta x_i = \Delta x_{i+1}$, we obtain:

$$a_{i+1} = a_{i-1} = 1$$

In this case

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x_i}$$

and we have the following estimate of the accuracy:

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{\partial f_i}{\partial x} + \frac{\Delta x^2{}_i}{6}\frac{\partial^3 f_i}{\partial x^3}$$

Let $N=4$, then

$$\frac{\partial \tilde{f}_i}{\partial x} \cdot 2(\Delta x_{i-1} + \Delta x_i) = a_{i+2}(f_{i+2} - f_i) + a_{i+1}(f_{i+1} - f_i) - a_{i-2}(f_{i-2} - f_i) - a_{i-1}(f_{i-1} - f_i)$$

After expanding variables $f_{i\mp 2}, f_{i\mp 1}$ in the Taylor series :

$$\frac{\partial \tilde{f}_i}{\partial x} \cdot 2(\Delta x_{i-1} + \Delta x_i) = \left[a_{i+2}(\Delta x_{i-1} + \Delta x_i) + a_{i+1}\Delta x_i + a_{i-2}(\Delta x_{i-1} + \Delta x_i) + a_{i-1}\Delta x_i\right] \cdot \frac{\partial f_i}{\partial x} +$$

$$+ \left[a_{i+2}(\Delta x_{i-1} + \Delta x_i)^2 + a_{i+1}\Delta x^2{}_i - a_{i-2}(\Delta x_{i-1} + \Delta x_i)^2 - a_{i-1}\Delta x^2{}_i\right] \cdot \frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} +$$

$$+ \left[a_{i+2}(\Delta x_{i-1} + \Delta x_i)^3 + a_{i+1}\Delta x^3{}_i + a_{i-2}(\Delta x_{i-1} + \Delta x_i)^3 + a_{i-1}\Delta x^3{}_i\right] \cdot \frac{1}{6}\frac{\partial^3 f_i}{\partial x^3} +$$

$$+ \left[a_{i+2}(\Delta x_{i-1} + \Delta x_i)^4 + a_{i+1}\Delta x^4{}_i - a_{i-2}(\Delta x_{i-1} + \Delta x_i)^4 - a_{i-1}\Delta x4_i\right] \cdot \frac{1}{24}\frac{\partial^4 f_i}{\partial x^4} +$$

$$+ \left[a_{i+2}(\Delta x_{i-1} + \Delta x_i)^5 + a_{i+1}\Delta x^5{}_i + a_{i-2}(\Delta x_{i-1} + \Delta x_i)^5 + a_{i-1}\Delta x^5{}_i\right] \cdot \frac{1}{120}\frac{\partial^5 f_i}{\partial x^5}$$

Hence, we find

$$a_{i+2}(\Delta x_{i-1} + \Delta x_i) + a_{i+1}\Delta x_i + a_{i-2}(\Delta x_{i-1} + \Delta x_i) + a_{i-1}\Delta x_i = 2(\Delta x_{i-1} + \Delta x_i)$$

$$a_{i+2}(\Delta x_{i-1} + \Delta x_i)^2 + a_{i+1}\Delta x^2{}_i - a_{i-2}(\Delta x_{i-1} + \Delta x_i)^2 - a_{i-1}\Delta x^2{}_i = 0$$

$$a_{i+2}(\Delta x_{i-1} + \Delta x_i)^3 + a_{i+1}\Delta x^3{}_i + a_{i-2}(\Delta x_{i-1} + \Delta x_i)^3 + a_{i-1}\Delta x^3{}_i = 0$$

$$a_{i+2}(\Delta x_{i-1} + \Delta x_i)^4 + a_{i+1}\Delta x^4{}_i - a_{i-2}(\Delta x_{i-1} + \Delta x_i)^4 - a_{i-1}\Delta x^4{}_i = 0$$

The main determinant of this system $\Delta_{04}^{(1)}$ is nonzero and is

$$\Delta^{(1)}{}_{04} = -4 \cdot \Delta x^2{}_{i-1} \cdot \Delta x^3{}_i (\Delta x_{i-1} + \Delta x_i)^3 (\Delta x_{i-1} + 2\Delta x_i)^2$$

Coefficient $a_{i\pm 2}$ and $a_{i\pm 1}$ satisfy the conditions of symmetry and equal:

$$a_{i+2} = a_{i-2} = -\frac{\Delta x^2{}_i}{\Delta x_{i-1}(\Delta x_{i-1} + 2\Delta x_i)}$$

$$a_{i+1} = a_{i-1} = -\frac{(\Delta x_{i-1} + \Delta x_i)^3}{\Delta x_{i-1} \cdot \Delta x_i (\Delta x_{i-1} + 2\Delta x_i)}$$

They depend on the number of steps because of its variability.

To calculate the $\frac{\partial \tilde{f}_i}{\partial x}$ we have the formula :

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{1}{2\Delta x_{i-1}\left(2 + \frac{\Delta x_{i-1}}{\Delta x_i}\right)} \left[\left(1 + \frac{\Delta x_{i-1}}{\Delta x_i}\right)^2 (f_{i+1} - f_{i-1}) - \frac{f_{i+2} - f_{i-2}}{1 + \frac{\Delta x_{i-1}}{\Delta x_i}} \right]$$

and the following expression to evaluate the accuracy :

$$\frac{\partial \tilde{f}_i}{\partial x} = \frac{\partial f_i}{\partial x} - \frac{\Delta x^4{}_i}{240}\left(1 + \frac{\Delta x_{i-1}}{\Delta x_i}\right)^2 \cdot \frac{\partial^5 f_i}{\partial x^5}$$

So for $N=4$ partial derivative $\frac{\partial \tilde{f}_i}{\partial x}$ calculated by the fourth-order accuracy.

Increasing N, we find the values of the coefficients $a_{i\pm m}$ at each point for any N.

Algorithm for calculating the value of the second partial derivative

In this calculation case we use the formula :

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{\tilde{n}\eth} = \sum_{m=1}^{N/2} \left[b_{i+m}(f_{i+m} - f_i) + b_{i-m}(f_{i-m} - f_i) \right],$$

where

$$(\Delta x_i)_{\tilde{n}\eth} = \sum_{\theta=-(\frac{N}{2}-1)}^{0} \Delta x_{i+\theta}$$

For $N=2$ $(\Delta x_i)_{cp} = \Delta x_i$, and

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}\Delta x_i^{\,2} = b_{i+1}(f_{i+1} - f_i) + b_{i-1}(f_{i-1} - f_i)$$

Expanding variables $b_{i\pm 1}$ in the Taylor series, we obtain:

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}\Delta x_i^{\,2} = (b_{i+1} - b_{i-1})\Delta x_i \frac{\partial f_i}{\partial x} + (b_{i+1} + b_{i-1})\Delta x^2{}_i \frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} + (b_{i+1} - b_{i-1})\Delta x^3{}_i \frac{1}{6}\frac{\partial^3 f_i}{\partial x^3} + (b_{i+1} + b_{i-1})\Delta x^4{}_i \frac{1}{24}\frac{\partial^4 f_i}{\partial x^4}$$

Hence, we find:

$$b_{i+1} - b_{i-1} = 0; \quad b_{i+1} + b_{i-1} = 2$$

Therefore $b_{i+1} = b_{i-1} = 1$

and $$\frac{\partial^2 \tilde{f}_i}{\partial x^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x_i^2}$$

Simultaneously, the coefficient of $\frac{\partial^3 f_i}{\partial x^3}$ is zero, and to evaluate the accuracy of calculation $\frac{\partial^2 f_i}{\partial x^2}$ we obtain the expression:

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2} = \frac{\partial^2 f_i}{\partial x^2} + \frac{\Delta x_i^2}{12} \cdot \frac{\partial^4 f_i}{\partial x^4}$$

Let N=4, then $(\Delta x_i)_{\text{ñð}} = \Delta x_{i-1} + \Delta x_i$, and

$$\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{\text{ñð}} = b_{i+2}(f_{i+2} - f_i) + b_{i+1}(f_{i+1} - f_i) + b_{i-2}(f_{i-2} - f_i) + b_{i-1}(f_{i-1} - f_i)$$

Expanding variables $b_{i\pm2}$ and $b_{i\pm1}$ in the Taylor series, we obtain :

$$\begin{aligned}
\frac{\partial^2 \tilde{f}_i}{\partial x^2}(\Delta x_i)^2_{\text{ñð}} = {} & \left[b_{i+2}(\Delta x_{i-1} + \Delta x_i) + b_{i+1}\Delta x_i - b_{i-2}(\Delta x_{i-1} + \Delta x_i) - b_{i-1}\Delta x_i \right] \frac{\partial f_i}{\partial x} + \\
& + \left[b_{i+2}(\Delta x_{i-1} + \Delta x_i)^2 + b_{i+1}\Delta x^2{}_i + b_{i-2}(\Delta x_{i-1} + \Delta x_i)^2 + b_{i-1}\Delta x^2{}_i \right] \frac{1}{2} \frac{\partial^2 f_i}{\partial x^2} + \\
& + \left[b_{i+2}(\Delta x_{i-1} + \Delta x_i)^3 + b_{i+1}\Delta x^3{}_i - b_{i-2}(\Delta x_{i-1} + \Delta x_i)^3 - b_{i-1}\Delta x^3{}_i \right] \frac{1}{6} \frac{\partial^3 f_i}{\partial x^3} + \qquad (*) \\
& + \left[b_{i+2}(\Delta x_{i-1} + \Delta x_i)^4 + b_{i+1}\Delta x^4{}_i + b_{i-2}(\Delta x_{i-1} + \Delta x_i)^4 + b_{i-1}\Delta x^4{}_i \right] \frac{1}{24} \frac{\partial^4 f_i}{\partial x^4} + \\
& + \left[b_{i+2}(\Delta x_{i-1} + \Delta x_i)^5 + b_{i+1}\Delta x^5{}_i - b_{i-2}(\Delta x_{i-1} + \Delta x_i)^5 - b_{i-1}\Delta x^5{}_i \right] \frac{1}{120} \frac{\partial^5 f_i}{\partial x^5} + \\
& + \left[b_{i+2}(\Delta x_{i-1} + \Delta x_i)^6 + b_{i+1}\Delta x^6{}_i + b_{i-2}(\Delta x_{i-1} + \Delta x_i)^6 + b_{i-1}\Delta x^6{}_i \right] \frac{1}{720} \frac{\partial^6 f_i}{\partial x^6}
\end{aligned}$$

Hence, we find :

$$b_{i+2}(\Delta x_{i-1}+\Delta x_i)+b_{i+1}\Delta x_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)-b_{i-1}\Delta x_i=0$$

$$b_{i+2}(\Delta x_{i-1}+\Delta x_i)^2+b_{i+1}\Delta x^2{}_i+b_{i-2}(\Delta x_{i-1}+\Delta x_i)^2+b_{i-1}\Delta x^2{}_i=2(\Delta x_{i-1}+\Delta x_i)^2$$

$$b_{i+2}(\Delta x_{i-1}+\Delta x_i)^3+b_{i+1}\Delta x^3{}_i-b_{i-2}(\Delta x_{i-1}+\Delta x_i)^3-b_{i-1}\Delta x^3{}_i=0$$

$$b_{i+2}(\Delta x_{i-1}+\Delta x_i)^4+b_{i+1}\Delta x^4{}_i+b_{i-2}(\Delta x_{i-1}+\Delta x_i)^4+b_{i-1}\Delta x^4{}_i=0$$

The main determinant of this system $\Delta_{04}^{(2)}$ is nonzero and equal $\Delta_{04}^{(1)}$

Coefficients $b_{i\pm 2}$ and $b_{i\pm 1}$

pairwise are equal and given by the expressions :

$$b_{i+2}=b_{i-2}=-\frac{\Delta x^2{}_i}{\Delta x_{i-1}(\Delta x_{i-1}+2\Delta x_i)}$$

$$b_{i+1}=b_{i-1}=-\frac{(\Delta x_{i-1}+\Delta x_i)^4}{\Delta x_{i-1}\cdot\Delta x^2{}_i(\Delta x_{i-1}+2\Delta x_i)}$$

Their values depend on the number of steps. To calculate the value of $\frac{\partial^2\tilde{f}_i}{\partial x^2}$

we obtain formula:

$$\frac{\partial^2\tilde{f}_i}{\partial x^2}=-\frac{\Delta x^2{}_i}{\Delta x_{i-1}(\Delta x_{i-1}+2\Delta x_i)}\frac{f_{i+2}-2f_i+f_{i-2}}{(\Delta x_{i-1}+\Delta x_i)^2}+\frac{(\Delta x_{i-1}+\Delta x_i)^4}{\Delta x_{i-1}\Delta x^2{}_i(\Delta x_{i-1}+2\Delta x_i)}\frac{f_{i+1}-2f_i+f_{i-1}}{(\Delta x_{i-1}+\Delta x_i)^2}$$

and the expression for estimating the accuracy of calculation $\frac{\partial^2 f_i}{\partial x^2}$ based on the $(N+2)$ -th term in the expansion (*):

$$\frac{\partial^2\tilde{f}_i}{\partial x^2}=\frac{\partial^2 f_i}{\partial x^2}-\Delta x^4{}_i\frac{\left(1+\frac{\Delta x_{i-1}}{\Delta x_i}\right)^2}{360}\frac{\partial^6 f_i}{\partial x^6}$$

We see that for $N = 4$ value $\frac{\partial^2 f_i}{\partial x^2}$ calculated by the fourth-order accuracy.

It should be noted that as for $N=2$ $(N+1)$-th term in the expansion (*) is zero.
In fact, since

$$b_{i+2} = b_{i-1} , \quad b_{i+1} = b_{i-1} ,$$

then the difference b_{i+2} - b_{i-2} and b_{i+1} - b_{i-1}

multiplied by $(\Delta x_{i-1} + \Delta x_i)^5$ and by Δx_i^5 , respectively, will be equal to zero.

If for $N=2$ and $N=4$ $(N+1)$-th term is equal to zero, then the following rule of mathematic induction the same result were obtained for any other N.

Increasing *N*, we obtain the coefficients $b_{i\pm m}$
for any value of N.

Conclusion

1. ***The expediency of using nonmonotonic changing (with a minimum) of spatial integration steps in going through the shock of hydromechanical parameters (HMP).***
2. ***Integration steps Δ x (x) must comply with logistic curves. By varying the parameters of the logistic curves, including the value of minimum step, you can change the number of integration steps within the zone shock of HMP.***
3. ***Expressions are obtained for calculating the gradient and second derivatives of both descending and ascending branches of the curve $\Delta x(x)$, and in its minimal point at any given value of the parameter N.***

Thank you for your time!